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INDEPENDENCE SATURATION OF SPLITTING GRAPHS

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Abstract. The independence number of a graph is one of the basic numerical characteristics of a graph. Additionally, it is one of the most fundamental and well-studied graph parameters. It is also used the proof the computational complexity of many theoretical problems. Thus, many independence-type parameters have been studied in the literature. The independence saturation number $IS(G)$ is one of the fundamental parameters introduced by Subramanian (Arumugam & Subramanian, 2007). In this paper, we examine the independence saturation numbers for splitting graph $S(G)$ when G is a specific type of graphs is computed, and exact formulae are derived.

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1 Introduction

This paper aims to research a simple undirected loopless graph with no isolated vertices. Let us consider the graph of $G = (V, E)$, where V is the vertex set and E is the edge set.

Domination, covering, and independence in graphs are topics covered in various graph-theoretic notions (Aytac & Turaci, 2011; Tuncel et al., 2015).

The set $D \subseteq V$ is defined as a dominating set of G, if each vertex in $V - D$ is attached to some vertex in D. Therefore, the domination number $\gamma(G)$ of G can be expressed as the minimum cardinality of a dominating set of G (Haynes et al., 1998).

A set of vertices known as the vertex cover of a graph is one that contains at least one endpoint for each edge. Finding a minimum vertex cover is a classic optimization issue in computer science. Numerous theoretical and practical issues can be modeled using the vertex cover optimization technique. For instance, the vertex cover minimization problem can be modeled as a commercial establishment interested in installing the fewest possible closed circuit cameras that cover all hallways (edges) connecting all rooms (nodes) on a floor. This problem also can be used to model the elimination of repetitive DNA sequences for synthetic biology and metabolic engineering applications (Hossain et al., 2020).

An independent set is described as a subset of the pairwise nonadjacent vertices in a G graph. The independence number of G is the cardinality of a maximum size independent set in G, and it is represented by the symbol $\beta(G)$. Many theoretical problems' computational complexity is demonstrated using the maximum independent set and its complement, the minimum vertex cover problem (Karp, 1972). Additionally, they can be used as models for real-world optimization issues (Pardalos & Xue, 1994; Liu et al., 2015). For instance, the largest independent set can be used to find stable genetic elements for the creation of designed genetic systems.

The independence number is the most fundamental parameter and one of the basic numerical characteristics of a graph, which is well-studied in the literature (Butenko et al., 2002; Araujo et al., 2011; Joo et al., 2016; Aytac & Turaci, 2009, 2012). To review the concerning algorithms, applications, and complexity issues of this problem, we refer to (Berberler & Berberler, 2017). The independence saturation number is one of the key concepts introduced by Subramanian among the independence-type characteristics that have been investigated (Arumugam & Subramanian, 2007).

The maximum cardinality of an independent set in a graph G that contains a vertex v is indicated by the notation $IS(v)$. The $\min\{IS(v) : v \in V\}$ is known as the independence saturation number of G and denoted by $IS(G)$. Therefore, $IS(G)$ is the biggest positive integer k such that each vertex of G is a member of an independent set of size k. If for a vertex $v \in V, IS(v) = IS(G)$, then any independent set with cardinality $IS(G)$ containing v is an IS-set. The issue of establishing whether $IS(G) \geq k$ is NP-complete for any graph G. Calculated in (Muthulakshmi & Subramanian, 2014) are the independence saturation numbers for several classes of graphs.

Let $G = (V, E)$ be a graph. The length (number of edges) of the shortest path between any two graph vertices is known as their distance. The largest distance between any two vertices is the diameter of the graph. We denote the diameter of G by diam (G). The degree of u is denoted by deg(u). The set of neighbors of u is denoted by $N(u)$. The closed neighborhood of the vertex u is denoted by $N[u] = N(u) \cup \{u\}$ (Chartrand & Lesniak, 2005; West, 1996).

The paper proceeds as follows. Firstly, we reviewed some known results of existing literature on independence saturation numbers. Then, the independence saturation numbers for splitting graph $S(G)$ when G is a specific type of graphs is computed and exact formulae are derived.

Theorem 1. *(Arumugam & Subramanian, 2007) If G is an r-regular graph on n vertices with* $\mathrm{r} > 0$*, then* $\mathrm{IS}(G) \leq n/2$*. Further equality holds if and only if G is bipartite.*

Theorem 2. *(Arumugam & Subramanian, 2007) The independence saturation of*

a. the cycle graph $C_n, \mathrm{IS}\left(C_n\right) = \lfloor n/2 \rfloor$.

b. the complete graph $K_n, IS\left(K_n\right) = 1$.

c. the complete bipartite graph $K_{m,n}, IS\left(K_{m,n}\right) = \min\{m, n\}$.

d. the star graph $K_{1,n}, IS\left(K_{1,n}\right) = 1$

Theorem 3. *(Berberler & Berberler, 2018) The independence saturation of*

a. the path $P_n (n \geq 2)$ *is* $\lfloor n/2 \rfloor$,

b. the wheel W_n *is 1 ,*

c. the comet $C_{t,r}$ *is* $\lceil t/2 \rceil$.

Corollary 1. *(Berberler & Berberler, 2017) If a vertex u has eccentricity one in graph G, then* $IS(u) = 1$.

Corollary 2. *(Berberler & Berberler, 2017) Let G be a graph with n vertices. If G has a vertex with eccentricity one, then* $IS(G) = 1$.

Theorem 4. *(Aytaç & Gürsan, 2019) The independence saturation of*

a. the $G \cong E_p^t$ *tree graph,* $IS(G) = t\lfloor p/2 \rfloor + 1$.

b. the generalized caterpillar graph $G \cong C_{(m,0)}P_n, IS(G) = (n-3)m + 2$.

c. the double star graph $G \cong S_{a,b}, IS(G) = \min\{a, b\} + 1$.

d. the binomial tree $G \cong B_n$*, for* $n \geq 1 \quad IS(G) = 2^{n-1}$.

e. the double comet $G \cong DC(n, a, b), IS(G) = \lfloor (n - a - b)/2 \rfloor + \min\{a, b\}$.

2 Independence Saturation of Splitting Graph

We begin this subsection by determining the independence saturation number of the splitting graph $S(G)$ in which G is a specific type of graphs. The splitting graph is important in graph theory. Therefore, there are many studies on the splitting graph in the literature (Turacı & Aytaç, 2017; Aytaç & Turacı, 2018).

Definition 1. *(Vaidya & Kothari, 2013) For a graph G, the splitting graph $S(G)$ of graph G is obtained by adding a new vertex corresponding to each vertex v of G such that $N(v) = N(v')$, where $N(v)$ and $N(v')$ are the neighborhood sets of v and v', respectively.*

Let G be a graph of order n and $V(G) = \{v_1, v_2, \ldots, v_n\}$. For splitting graph of G of order $2n$, let $V(S(G)) = X \cup Y$, where $X = \{v_1, v_2, \ldots, v_n\}, Y = \{v_1', v_2', \ldots, v_n'\}$.

Theorem 5. *Let $G \cong K_n$ be a complete graph. Then $IS(S(G)) = 2$.*

Proof. The vertex set of $S(G)$ can be partitioned into two subset as

$$V_1 = \{v = v_i \mid v \in V(G), 1 \leq i \leq n\}$$

and

$$V_2 = \left\{v' = v_j' \mid v' \in V(S(G)) - V(G), 1 \leq j \leq n\right\}.$$

Two cases exist according to the vertices of $S(G)$.

Case 1. If every v_i is in V_1, then v_i is adjacent to all of vertices except the vertex $v_j' \in V_2$ where $j = i$ by the definition of $S(G)$. Thus, the set $\{v_i\} \cup \left\{v_j'\right\}$ is the $IS(v_i)-$ set of $S(G)$. Hence, it is obtained $\mathrm{IS}(v_i) = 2$.

Case 2. If every v_j' is in V_2, then $N_{S(G)}\left(v_j'\right) = \bigcup_{\substack{i=1 \\ i \neq j}}^{n} v_i$ and $\left|N_{S(G)}\left(v_j'\right)\right| = n - 1$. It is not adjacent to the graph $S(G)$ containing the vertex v_j' is a $n-1+1 = n$ vertex set. Then it follows $IS\left(v_j'\right) = n$

From *Case 1* and *2*, we have $IS(S(G)) = \min\{n, 2\} = 2$.
Thus, the proof holds. □

Theorem 6. *Let $G \cong C_n$ be a cycle graph. Then $IS(S(G)) = 2\left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. Have a vertex partitioning in the form $V(S(G)) = V_1 \cup V_2$, where

$$V_1 = \{v = v_i \mid v \in V(G), 1 \leq i \leq n\}$$

and

$$V_2 = \left\{v' = v_j' \mid v' \in V(S(G)) - V(G), 1 \leq j \leq n\right\}.$$

Two cases exist according to the vertices of $S(G)$.

Case 1. Let $v \in V_1$. Since the graph C_n is a vertex-transitive graph, the value IS(v) is the same for all $v \in V_1$. Let I be a $IS(v)$-set containing the vertex $v.I = I_1 \cup I_2$ where $I_1 = \{v \mid \exists v \in V_1\}$ and $I_2 = \{v' \mid \exists v' \in V_2\}$

If the discussed vertex is a single index vertex, then it is seen that the set I_1 consists of nonadjacent single index vertices of the graph C_n. If the discussed vertex is a double index vertex, then it is seen that the set I_1 consists of non-adjacent double index vertices of the graph C_n. Then it is seen that the acquired set I_1 corresponds to the independence number of the graph C_n as well. In other words, it is obtained $|I_1| = \left\lfloor \frac{n}{2} \right\rfloor$. The vertices of the set I_2 are copies of the vertices of the set I_1 in $S(G)$. Consequently, $|I_1| = |I_2|$. Hence $I = 2I_1 = 2I_2$ for all $v \in V_1$. This implies $IS(v) = 2\left\lfloor \frac{n}{2} \right\rfloor$.

Case 2. If $v' \in V_2$, then every vertex $v' \in V_2$ is adjacent to only 2 vertices in the set V_1. The vertices constituting the set V_2 are not adjacent to each other. Let I be a $IS\,(v')$-set for the vertex v'. It is clear that the set I consists of only the set V_2. Then $I = \bigcup_{i=1}^{n} v_i'$. Thus, it is obtained $IS\,(v') = n$.

From *Case 1* and *2*, we have $IS(G) = \min\left\{2\left\lfloor\frac{n}{2}\right\rfloor, n\right\} = 2\left\lfloor\frac{n}{2}\right\rfloor$.
This completes the proof. □

Theorem 7. *Let $G \cong K_{m,n}$ be a complete bipartite graph. Then $IS(S(G)) = 2\min\{m,n\}$.*

Proof. The vertex set of $S(G)$ can be partitioned into two subset as $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$ in which

$$V_1 = \{v = v_i \mid 1 \le i \le m\}, \quad V_2 = \{v = v_i \mid m+1 \le i \le m+n\}, V_3 = \left\{v' = v_i' \mid 1 \le i \le m\right\}$$

and

$$V_4 = \left\{v' = v_i' \mid m+1 \le i \le m+n\right\}.$$

The vertices in the set V_3 are copies of the vertices in V_1, and the vertices in the set V_4 are copies of the vertices in V_2. There are four different cases according to the vertices of the graph $S(G)$.

Case 1. If $v_i \in V_1$, then $N_{S(G)}\,(v_i) = \bigcup_{i=m+1}^{m+n} \{v_i; v'\}$. Also, $\left|N_{S(G)}\,(v_i)\right| = 2n$. Let the set I be a IS (v_i) - set of the graph $S(G)$. Since the vertices in the set $N_G\,(v_i)$ consists of the vertices in V_2 and V_4, the remainder vertices in the graph belong to the sets V_1 and V_3. The vertices in the set V_1 are not adjacent to each other by the structure of the graph $K_{m,n}$. Also, the vertices in the set V_3 are copies of those of V_1, and they are not adjacent to the vertices of V_1 by the definition of splitting graph. Then the vertices in V_1 and V_3 are not adjacent to each other. Hence, $I = \bigcup_{i=1}^{m} \{v_i; v_i'\}$, and so IS $(v_i) = 2(m-1+1) = 2m$.

Case 2. If $v_i \in V_2$, then $N_{S(G)}\,(v_i) = \bigcup_{i=1}^{m} \{v_i; v_i'\} = V_1 \cup V_3$. Let the set I be a $IS\,(v_i) -$ set of the graph $S(G)$. Because the vertices in the set $N_G\,(v_i)$ consists of the vertices V_1 ve V_3, the remainder vertices in the graph belong to the sets V_2 and V_4. The vertices in the set V_2 are not adjacent to each other by the structure of the graph $K_{m,n}$. Also, the vertices in the set V_4 are copies of those of V_2, and they are not adjacent to the vertices of V_2 by the definition of splitting graph. Then the vertices in V_2 and V_4 are not adjacent to each other. Thus, $I = \bigcup_{i=m+1}^{m+n} \{v_i; v_i'\}$, and so $IS\,(v_i) = 2(m+n-(m+1)+1) = 2n$

Case 3. If $v_i \in V_3$, then $N_{S(G)}\,(v_i) = \bigcup_{i=1}^{m} \{v_i; v_i'\} = V_2 \cup V_4$. Let the set I be a IS $(v_i) -$ set of the graph $S(G)$. Since the vertices in the set $N_G\,(v_i)$ consists of the vertices in V_2 and V_4, the remainder vertices in the graph belong to the sets V_1 and V_3. The vertices in the set V_1 are not adjacent to each other by the structure of the graph $K_{m,n}$. Also, the vertices in the set V_3 are copies of those of V_1, and they are not adjacent to the vertices of V_1 by the definition of splitting graph. Then the vertices in V_1 and V_3 are not adjacent to each other. Therefore, $I = \bigcup_{i=m+1}^{m+n} \{v_i; v_i'\}$, and so IS $(v_i) = 2(m-1+1) = 2m$

Case 4. If $v_i \in V_4$, then $N_{S(G)}\,(v_i) = \bigcup_{i=1}^{m} \{v_i; v_i'\} = V_1 \cup V_3$. Let the set I be a IS (v_i) - set of the graph $S(G)$. Because the vertices in the set $N_G\,(v_i)$ consists of the vertices V_1 ve V_3, the remainder vertices in the graph belong to the sets V_2 and V_4. The vertices in the set V_2 are not adjacent to each other by the structure of the graph $K_{m,n}$. Also, the vertices in the set V_4 are copies of those of V_2, and they are not adjacent to the vertices of V_2 by the definition of splitting graph. Thereby, $I = \bigcup_{i=m+1}^{m+n} \{v_i; v_i'\}$, and so, IS $(v_i) = 2(m+n-(m+1)+1) = 2n$.

From *Case 1,2,3* and *4*, $IS(S(G)) = \min\{2m, 2n\} = 2\min\{m, n\}$.
Hence, the proof holds.

$\square$

Theorem 8. *Let $G \cong P_n$ be a path graph. Then $IS(S(G)) = 2\left\lceil\frac{n-1}{2}\right\rceil$.*

Proof. The vertex set of $S(G)$ can be partitioned into two subset as $V_1 = \{v_i \in V \mid 1 \le i \le n\}, V_2 = \{v'_i \in V \mid 1 \le i \le n\}$. The vertices in the set V_2 consist of copies of the vertices in the set V_1, and $N_G(v_i) = N_{S(G)}(v_i)$ where $v_i \in V_1, v'_i \in V_2 (1 \le i \le n)$. There are two cases according to the vertices of the graph G.

Case 1. If $v_i \in V_1$ is a single index vertex, then I consists of the single index vertices in the set V_1 by the structure of the graph P_n and copies of these vertices in the set V_2 by the definition of splitting graph such that I is a $IS(v_i)$ - set of the graph G. Since the number of single index vertices in the set V_1 is $\left\lceil\frac{n}{2}\right\rceil$, the number of their copies in the set V_2 is $\left\lceil\frac{n}{2}\right\rceil$. Consequently, IS $(v_i) = 2\left\lceil\frac{n}{2}\right\rceil$.

If v_i is a double index vertex, then I consists of the double index vertices in the set V_1 by the structure of the graph P_n and copies of these vertices in the set V_2 by the definition of splitting graph. Because the number of double index vertices in the set V_1 is $\left\lceil\frac{n-1}{2}\right\rceil$, the number of their copies in the set V_2 is $\left\lceil\frac{n-1}{2}\right\rceil$. Thus, IS $(v_i) = 2\left\lceil\frac{n-1}{2}\right\rceil$.

Case 2. If $v'_i \in V_1$, then the vertices constituting the set I consist of the vertex being the equivalent of the vertex v'_i in the set V_1 and the vertices which are not adjacent to each other in the set V_2 by the definition of splitting graph such that I is a $IS(v_i)$ - set of the graph G. Hence, $I = \{v'_i \mid v_i \in V_1\} \cup V_2$. Then $IS(v'_i) = 1 + n$.

From *Case 1* and *2*, $IS(G) = \min\left\{2\left\lceil\frac{n}{2}\right\rceil, 2\left\lceil\frac{n-1}{2}\right\rceil, 1+n\right\} = 2\left\lceil\frac{n-1}{2}\right\rceil$.
Thus, the proof holds. $\square$

Theorem 9. *If the graph G contains a $(n-1)$ degree vertex, then* $\mathrm{IS}(S(G)) = 2$.

Proof. Let v_1 be a $(n-1)$ degree vertex of the graph G. It is easily seen that $N_{S(G)}(v_1) = \{\bigcup_{i=2}^{n} v_i\} \cup \{\bigcup_{i=2}^{n} v'_i\}$. Then it follows $deg_{S(G)}(v_1) = 2n-2$. In the graph $S(G)$, this is also the value $\Delta(S(G))$. Because the independence saturation value of a graph is minimum, it is sufficient to look just in the IS $(v_1)-$ set for the vertex v_1. The set $\{v_1\} \cup \{v'_1\}$ is a $IS(v_1)-$ set. Therefore, $IS(v_1) = 2$. This provides $\mathrm{IS}(S(G)) = 2$.
Then, the proof is completed. $\square$

As a result of Theorem 9 , the following result is obtained.

Corollary 3. *If a graph G is one of the graphs $S_{1,n}, W_{1,n}$ and K_n, then $IS(S(G)) = 2$.*

3 Conclusion

For any graph G we have investigated how the independence saturation value for splitting graph $S(G)$ is related. The results reported here throw some light in the direction to find the independence saturation value of larger graph obtained from the given graph. A similar study for other graph theoretic parameters is an interesting direction for further research.

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